

On a Theorem in the Geometry of Numbers in a Space of Laurent Series

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Proof of a general inequality connecting point sets with lattices in a space of Laurent series.

In 1948 Chabauty [1] and Rogers [2] simultaneously and independently proved the inequality

$$\Delta(S) \prod_{k=1}^n m_k(S, A) \leq 2^{(n-1)/2} d(A)$$

for the successive minima $m_k(S, A)$ of an arbitrary point set S relative to any lattice A in real n -dimensional space R^n . Both Chabauty [1] and Mahler [3] proved independently that the constant factor $2^{(n-1)/2}$ is best possible.

We shall now establish the analogous inequality

$$\Delta(S) \prod_{k=1}^n m_k(S, A) \leq 2^n d(A)$$

for point sets S and lattices A in the n -dimensional space F^n with coordinates in a field F of formal Laurent series in one indeterminate. It will, however, be assumed that

$$0 < \Delta(S) < \infty \quad \text{and} \quad 0 < m_k(S, A) < \infty \quad (k = 1, 2, \dots, n).$$

The proof is similar to that by Rogers in the real case, but since the distance in F^n arises from a discrete non-archimedean valuation, certain changes are necessary. The quantities $\Delta(S)$, $m_k(S, A)$, and $d(A)$ will be defined in the text. Again the constant factor 2^n will be shown to be best possible.

1

Let f be any field, z an indeterminate, $f[z]$ the ring of polynomials and $f(z)$ the field of rational functions in z with coefficients in f .

A special non-archimedean valuation $|x|$ on the field $f(z)$ is given by¹

$$|0| = 0; \quad |x| = 2^{N-D} \quad \text{for } x \neq 0.$$

Here N and D are the degrees of the numerator and of the denominator of the rational function x , respectively. This valuation is discrete, and any collection of its values greater than a positive constant contains a smallest element.

Let F be the completion of the field $f(z)$ relative to the valuation $|x|$. The elements x of F can be written as formal Laurent series

$$x = \sum_{j=-\infty}^r a_j z^j$$

with coefficients a_j in f . Here r may be any integer $< 0, = 0$, or > 0 . When $x = 0$, all coefficients a_j are zero; while for $x \neq 0$ the integer r will be chosen such that $a_r \neq 0$. The continuation of the valuation $|x|$ from $f(z)$ to F is now defined by

$$|0| = 0; \quad |x| = 2^r \quad \text{if } x \neq 0.$$

We define the *integral part* $[x]$ and the *fractional part* (x) of any element x of F by

$$[x] = \sum_{j=0}^r a_j z^j \quad \text{and} \quad (x) = \sum_{j=-\infty}^{-1} a_j z^j.$$

With this notation for every x in F ,

$$x = [x] + (x), \quad [x] \in f[z], \quad |(x)| \leq \frac{1}{2}.$$

2

Denote by n a fixed positive integer and by F^n the n -dimensional space of all points or vectors

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n), \quad \mathbf{0} = (0, \dots, 0),$$

¹ In Mahler [3], I defined the valuation $|x|$ instead by $|x| = e^{\Lambda - D}$.

etc., with coordinates (components) in F . The point $\mathbf{0}$ is the *origin of F^n* and plays a specialised role in the theory.

As usual the sum $\mathbf{x} + \mathbf{y}$ and the difference $\mathbf{x} - \mathbf{y}$ of two points \mathbf{x} and \mathbf{y} in F^n are defined by

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n) \quad \text{and} \quad \mathbf{x} - \mathbf{y} = (x_1 - y_1, \dots, x_n - y_n).$$

If further t is any element of F , then $t\mathbf{x}$ denotes the point

$$t\mathbf{x} = (tx_1, \dots, tx_n).$$

A norm $|\mathbf{x}|$ for the points \mathbf{x} in F^n is defined by

$$|\mathbf{x}| = \max(|x_1|, \dots, |x_n|).$$

It has the properties

$$|\mathbf{x} \pm \mathbf{y}| \leq \max(|\mathbf{x}|, |\mathbf{y}|), \quad |t\mathbf{x}| = |t| |\mathbf{x}|.$$

The expression

$$\rho(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$$

represents a distance in F^n and changes this space into a metric space, with all the usual topological implications.

3

The points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ in F^n are said to be *linearly dependent* (viz., over F) if there exists elements t_1, \dots, t_m of F not all zero such that

$$t_1 \mathbf{x}^{(1)} + \dots + t_m \mathbf{x}^{(m)} = \mathbf{0},$$

and they are otherwise called *linearly independent*. Further \mathbf{x} is said to be *linearly dependent on $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$* if elements s_1, \dots, s_m of F exist such that

$$\mathbf{x} = s_1 \mathbf{x}^{(1)} + \dots + s_m \mathbf{x}^{(m)}.$$

If no such s_1, \dots, s_m exist, then \mathbf{x} is called *linearly independent of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$* .

If the m points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly independent, and if the further point \mathbf{x} is linearly independent of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$, then the $m + 1$ points $\mathbf{x}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ are linearly independent.

A set of exactly n points

$$\mathbf{x}^{(k)} = (x_{k1}, \dots, x_{kn}) \quad (k = 1, 2, \dots, n)$$

in F^n is linearly independent if and only if the determinant

$$D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

is distinct from zero.

Any $n + 1$ points in F^n are linearly dependent.

4

Let S be an arbitrary point set in F^n . If t is any element of F , then tS denotes the set of all points

$$t\mathbf{x}, \quad \text{where } \mathbf{x} \in S.$$

The largest number of linearly independent points of S is called the *dimension of S* and denoted by $\delta(S)$. We put $\delta(S) = 0$ if S is empty or consists of the single point $\mathbf{0}$. In all other cases $\delta(S)$ is positive and satisfies

$$1 \leq \delta(S) \leq n.$$

It is obvious that

$$\delta(S) = \delta(tS) \quad \text{if } t \neq 0 \text{ lies in } F.$$

Of particular importance are two kinds of point sets, the linear manifolds M and the lattices \mathcal{A} .

A *linear manifold* M is a nonempty point set in F^n with the following two properties:

- (a) If $\mathbf{x} \in M$ and $\mathbf{y} \in M$, then also $\mathbf{x} + \mathbf{y} \in M$ and $\mathbf{x} - \mathbf{y} \in M$.
- (b) If $\mathbf{x} \in M$ and $t \in F$, then also $t\mathbf{x} \in M$.

The second property implies that all linear manifolds contain the origin $\mathbf{0}$. If M contains other points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$, then also all points

$$t_1 \mathbf{x}^{(1)} + \cdots + t_m \mathbf{x}^{(m)}, \quad \text{where } t_1, \dots, t_m \in F,$$

belong to M , and it is clear that $\delta(M) \geq 1$. There exist now $\delta(M) = \delta$ linearly independent points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\delta)}$ in M such that M consists exactly of the points

$$t_1 \mathbf{x}^{(1)} + \cdots + t_\delta \mathbf{x}^{(\delta)}, \quad \text{where } t_1, \dots, t_\delta \in F.$$

It follows that

$$tM = M \quad \text{for every } t \neq 0 \text{ in } F.$$

5

A lattice \mathcal{A} in F^n is a point set with the following four properties:

- (A) If $\mathbf{x} \in \mathcal{A}$ and $\mathbf{y} \in \mathcal{A}$, then also $\mathbf{x} + \mathbf{y} \in \mathcal{A}$ and $\mathbf{x} - \mathbf{y} \in \mathcal{A}$.
- (B) If $x \in \mathcal{A}$ and $u \in f[z]$, then also $ux \in \mathcal{A}$.
- (C) \mathcal{A} contains n linear independent points; thus $\delta(\mathcal{A}) = n$.
- (D) There exists a positive number c such that

$$|\mathbf{x}| \geq c \quad \text{for every point } \mathbf{x} \neq \mathbf{0} \text{ of } \mathcal{A}.$$

It follows from (B) that the origin $\mathbf{0}$ belongs to every lattice.

A set of n linearly independent points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of \mathcal{A} is called a *basis* of \mathcal{A} if every point \mathbf{x} of \mathcal{A} can be written in the form

$$\mathbf{x} = u_1 \mathbf{x}^{(1)} + \dots + u_n \mathbf{x}^{(n)}, \quad \text{where } u_1, \dots, u_n \in f[z].$$

LEMMA 1. *Every lattice has a basis.*

Proof. Choose for $\mathbf{x}^{(1)}$ a point of \mathcal{A} distinct from $\mathbf{0}$ such that $|\mathbf{x}^{(1)}|$ is as small as possible. Next, if $2 \leq k \leq n$ and if the points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$ of \mathcal{A} have already been selected, choose for $\mathbf{x}^{(k)}$ a point of \mathcal{A} which is linearly independent of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$ and has the property that $|\mathbf{x}^{(k)}|$ is a minimum.

The n lattice points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ so defined evidently satisfy

$$c \leq |\mathbf{x}^{(1)}| \leq |\mathbf{x}^{(2)}| \leq \dots \leq |\mathbf{x}^{(n)}|.$$

We assert that they form a basis of \mathcal{A} .

For consider an arbitrary point \mathbf{x} of \mathcal{A} . This point can certainly be written in the form

$$\mathbf{x} = t_1 \mathbf{x}^{(1)} + \dots + t_n \mathbf{x}^{(n)}, \quad \text{where } t_1, \dots, t_n \in F.$$

Put

$$\mathbf{x}^* = [t_1] \mathbf{x}^{(1)} + \dots + [t_n] \mathbf{x}^{(n)} \quad \text{and} \quad \mathbf{x}^{**} = (t_1) \mathbf{x}^{(1)} + \dots + (t_n) \mathbf{x}^{(n)}.$$

Then

$$\mathbf{x} = \mathbf{x}^* + \mathbf{x}^{**}$$

and

$$[t_k] \in f[z], \quad |(t_k)| \leq \frac{1}{2} \quad (k = 1, 2, \dots, n).$$

Therefore, by properties (A) and (B), \mathbf{x}^* and hence also \mathbf{x}^{**} belong to \mathcal{A} . The assertion will be proved if we can show that $\mathbf{x}^{**} = \mathbf{0}$, or equivalently,

$$(t_1) = \dots = (t_n) = 0.$$

If this were false, there would exist a suffix k such that

$$(t_k) \neq 0, \quad (t_{k+1}) = \dots = (t_n) = 0,$$

so that \mathbf{x}^{**} would be distinct from $\mathbf{0}$ for $k = 1$, but would be linearly independent of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$ if $k \geq 2$. However,

$$|\mathbf{x}^{**}| = |(t_1)\mathbf{x}^{(1)} + \dots + (t_k)\mathbf{x}^{(k)}| \leq 2^{-1} \max(|\mathbf{x}^{(1)}|, \dots, |\mathbf{x}^{(k)}|) < |\mathbf{x}^{(k)}|,$$

contrary to the definition of $\mathbf{x}^{(k)}$.

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With every basis $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of \mathcal{A} we can associate the determinant $D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})$; we know that it does not vanish.

LEMMA 2. *If $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ and $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}$ are two bases of \mathcal{A} , then*

$$|D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})| = |D(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)})|.$$

Proof. There exist two $n \times n$ matrices $(u_{hk})_{h,k=1,2,\dots,n}$ and $(v_{hk})_{h,k=1,2,\dots,n}$ with elements u_{hk}, v_{hk} in $f[z]$ such that

$$\mathbf{y}^{(h)} = u_{h1}\mathbf{x}^{(1)} + \dots + u_{hn}\mathbf{x}^{(n)} \quad \text{and} \quad \mathbf{x}^{(h)} = v_{h1}\mathbf{y}^{(1)} + \dots + v_{hn}\mathbf{y}^{(n)} \\ (h, k = 1, 2, \dots, n).$$

Denote their determinants by U and V , respectively. Then

$$D(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}) = U \cdot D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}), \\ D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = V \cdot D(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}),$$

from which it follows that

$$UV = 1.$$

Since U and V are polynomials in $f[z]$, U and V necessarily lie in f and are not equal to zero, hence satisfy

$$|U| = |V| = 1.$$

From this the assertion follows immediately.

The lemma implies that if we put

$$d(A) = |D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})|,$$

then $d(A)$ does not depend on the special basis $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of A . We call $d(A)$ the *determinant of A* . This determinant is always positive and equal to an integral power of 2.

7

From now on pairs (S, A) of a point set S and a lattice A in F^n will be considered. The lattice is said to be *S -admissible* if none of its points distinct from $\mathbf{0}$ belongs to S . If there are no S -admissible lattices, write

$$\Delta(S) = \infty.$$

Otherwise define $\Delta(S)$ as the lower bound

$$\Delta(S) = \inf d(A)$$

extended over all S -admissible lattices A .

Possibly $\Delta(S)$ is equal to 0; this happens when there are S -admissible lattices A with arbitrarily small determinant $d(A)$.

There remains the case when

$$0 < \Delta(S) < \infty.$$

Now the lower bound $\Delta(S)$ is attained, and there is at least one S -admissible lattice A satisfying

$$d(A) = \Delta(S).$$

Such a lattice is called a *critical lattice of S* .

8

We continue to consider the pair (S, A) of a point set S and a lattice A in F^n . Let k run over the integers from 1 to n . Denote by \mathcal{E}_k the set of all

integers N for which the set $z^N S$ contains k linearly independent points of A . It is obvious that

$$\Sigma_1 \supset \Sigma_2 \supset \cdots \supset \Sigma_n. \quad (1)$$

If Σ_k is empty, write

$$N_k = +\infty \quad \text{and} \quad m_k(S, A) = 2^{N_k} = +\infty.$$

If Σ_k contains a sequence of integers tending to $-\infty$, write

$$N_k = -\infty \quad \text{and} \quad m_k(S, A) = 2^{N_k} = 0.$$

If neither of these two cases holds, then Σ_k contains a smallest integer, N_k say. Now put

$$m_k(S, A) = 2^{N_k},$$

so that

$$0 < m_k(S, A) < \infty.$$

In this third case the set $z^{N_k} S$, but not the set $z^{N_k-1} S$, contains k linearly independent points of A .

It is clear from (1) that always

$$-\infty \leq N_1 \leq N_2 \leq \cdots \leq N_n \leq +\infty, \quad (2)$$

and therefore

$$0 \leq m_1(S, A) \leq m_2(S, A) \leq \cdots \leq m_n(S, A) \leq \infty. \quad (3)$$

The numbers $m_k(S, A)$ are called *the successive minima of S in A* .

9

Assume now that all the integers N_k are finite, hence that

$$\begin{aligned} -\infty < N_1 \leq N_2 \leq \cdots \leq N_n < +\infty, \\ 0 < m_1(S, A) \leq m_2(S, A) \leq \cdots \leq m_n(S, A) < \infty. \end{aligned} \quad (4)$$

For each $k = 1, 2, \dots, n$ form the point set

$$I_k = (z^{N_k-1} S \cap A) \cup \{\mathbf{0}\}$$

which consists of the origin $\mathbf{0}$ and of all the points of $z^{N_k-1} S$ which lie in A .

As was mentioned, the set $z^{N_k-1}S$ cannot contain more than $k-1$ points of \mathcal{A} which are linearly independent.

Denote by M_k the linear manifold in F^n which is spanned by the points of I_k . Its dimension $\delta(M_k) = \delta_k$ satisfies the inequality

$$0 \leq \delta_k \leq k-1.$$

Choose an arbitrary $\mathbf{x} \neq \mathbf{0}$ of I_k if such a point exists; this point lies of course also in M_k . Since $N_{k+1} \geq N_k$ and since therefore $z^{N_{k+1}-N_k} \in f[z]$,

$$z^{N_{k+1}-N_k}\mathbf{x} \in \mathcal{A}.$$

Further $\mathbf{x} \in z^{N_k-1}S$ and therefore

$$z^{N_{k+1}-N_k}\mathbf{x} \in z^{N_{k+1}-N_k} \cdot z^{N_k-1}S = z^{N_{k+1}-1}S.$$

It follows that

$$z^{N_{k+1}-N_k}\mathbf{x} \in I_{k+1}.$$

This relation implies that

$$z^{N_{k+1}-N_k}M_k \subset M_{k+1}.$$

On the other hand M_k is a linear manifold and therefore

$$z^{N_{k+1}-N_k}M_k = M_k.$$

Therefore $M_k \subset M_{k+1}$ and hence

$$M_1 \subset M_2 \subset \dots \subset M_n.$$

From this set of relations we can deduce that \mathcal{A} has a basis $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ such that the manifolds M_k is for every k spanned by the points

$$\mathbf{0}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\delta_k)}.$$

10

From this basis $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ of \mathcal{A} we derive now a new lattice \mathcal{A}^* with the basis

$$\mathbf{X}^{(1)} = z^{-N_1+1}\mathbf{x}^{(1)}, \mathbf{X}^{(2)} = z^{-N_2+1}\mathbf{x}^{(2)}, \dots, \mathbf{X}^{(n)} = z^{-N_n+1}\mathbf{x}^{(n)}$$

and therefore with the determinant

$$d(\mathcal{A}^*) = 2^{-N_1-N_2-\dots-N_n+n} d(\mathcal{A}). \quad (5)$$

LEMMA 3. *The lattice Λ^* is S -admissible.*

Proof. Let the assertion be false, i.e., let there exist a point $\mathbf{X} \neq \mathbf{0}$ which belongs to both S and Λ^* . In terms of the basis of Λ^* ,

$$\mathbf{X} = u_1 \mathbf{x}^{(1)} + \cdots + u_n \mathbf{x}^{(n)},$$

where u_1, \dots, u_n are polynomials not all zero in $f[z]$. Assume, say, that

$$u_k \neq 0, \quad u_{k+1} = u_{k+2} = \cdots = u_n = 0.$$

The new point

$$\mathbf{Y} = z^{N_k-1} \mathbf{X} = z^{N_k-N_1} u_1 \mathbf{x}^{(1)} + z^{N_k-N_2} u_2 \mathbf{x}^{(2)} + \cdots + u_k \mathbf{x}^{(k)}$$

belongs to the original lattice Λ because the coefficients

$$z^{N_k-N_1} u_1, z^{N_k-N_2} u_2, \dots, u_k$$

lie by $N_1 \leq N_2 \leq \cdots \leq N_n$ in the ring $f[z]$. It is further obvious that \mathbf{Y} belongs to the set $z^{N_k-1} S$. Hence \mathbf{Y} is a point of I_k and so also a point of M_k . However, by $u_k \neq 0$, the point \mathbf{Y} is linearly independent of the $k-1$ basis points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$, from which by $\delta_k \leq k-1$ it follows that \mathbf{Y} is also linearly independent of the points $\mathbf{0}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\delta_k)}$ which span the linear manifold M_k . Hence a contradiction arises, proving the assertion.

The main result of this paper follows now at once.

THEOREM 1. *If the set S and the lattice Λ satisfy condition (4), then the following inequality holds:*

$$\Delta(S) \prod_{k=1}^n m_k(S, \Lambda) \leq 2^n d(\Lambda).$$

Proof. Lemma 3 implies that

$$d(\Lambda^*) \geq \Delta(S).$$

On substituting here for $d(\Lambda^*)$ its value (5) and noting that $2^{N_k} = m_k(S, \Lambda)$ for all k , the assertion is obtained immediately.

Theorem 1 shows in particular that if (4) holds, then $\Delta(S) < \infty$. In fact, it suffices to assume that N_1 and therefore also $m_1(S, \Lambda)$ are finite. For then no point of Λ distinct from $\mathbf{0}$ lies in $z^{N_1-1} S$ so that Λ is $z^{N_1-1} S$ -admissible. Hence the lattice $z^{-N_1+1} \Lambda$ is S -admissible and therefore $\Delta(S) < \infty$.

11

It is obvious that every bounded set S has admissible lattices and hence satisfies $\Delta(S) < \infty$. There is also no difficulty in constructing unbounded sets with the same property.

The successive minima $m_k(S, A)$ depend on both the set S and the lattice A and they may well be equal to ∞ . Thus, when S lies in a linear manifold M of dimension $\delta \leq n-1$, then clearly for every lattice A ,

$$m_{\delta+1}(S, A) = m_{\delta+2}(S, A) = \cdots = m_n(S, A) = \infty.$$

However, the following simple result holds.

THEOREM 2. *If S has at least one interior point, then*

$$m_k(S, A) < \infty \quad (k = 1, 2, \dots, n)$$

for all lattices A .

Proof. To the interior point of S , \mathbf{X} say, there exists a positive number C such that all point \mathbf{x} in F^n satisfying

$$|\mathbf{x} - \mathbf{X}| \leq C$$

belong to S . Hence, for every integer N , all points \mathbf{x} in F^n satisfying

$$|\mathbf{x} - z^N \mathbf{X}| \leq 2^N C$$

lie in the set $z^N S$.

Denote now by $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ a basis of A and put

$$c = \max(|\mathbf{x}^{(1)}|, \dots, |\mathbf{x}^{(n)}|);$$

further choose N so large that

$$2^N C \geq c.$$

We assert then that the set $z^N S$ contains n linearly independent points of A , so that the assertion holds. For just as in the proof of Lemma 1 we can find first a point $\mathbf{x}(0)$ of A satisfying

$$|\mathbf{x}(0) - 2^N \mathbf{X}| \leq c/2,$$

and then the n further points $\mathbf{x}(k)$ of A defined by

$$\mathbf{x}(k) = \mathbf{x}(0) + \mathbf{x}^{(k)} \quad (k = 1, 2, \dots, n)$$

have the property

$$|\mathbf{x}(k) - z^N \mathbf{X}| \leq c \quad (k = 1, 2, \dots, n).$$

All $n + 1$ lattice points $\mathbf{x}(0), \mathbf{x}(1), \dots, \mathbf{x}(n)$ lie therefore in $z^N S$. But since

$$\mathbf{x}^{(k)} = \mathbf{x}(k) - \mathbf{x}(0) \quad (k = 1, 2, \dots, n),$$

certain n of these lattice points are linearly independent.

I know of no similar simple condition under which one or more of the successive minima $m_k(S, A)$ are distinct from zero. This problem is for unbounded sets S connected to quite difficult questions in Diophantine approximations.

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Let us finally prove that the factor 2^n in inequality (I) is best possible. The proof is an immediate consequence of

THEOREM 3. *The set S of all points $\mathbf{x} = (x_1, \dots, x_n)$ satisfying*

$$|\mathbf{x}| = \max(|x_1|, \dots, |x_n|) \leq 1$$

has the property that

$$A(S) = 2^n. \quad (6)$$

Let us for the moment assume that this theorem is true, and choose for A the lattice with the basis

$$\mathbf{e}^{(1)} = (1, 0, \dots, 0), \mathbf{e}^{(2)} = (0, 1, \dots, 0), \dots, \mathbf{e}^{(n)} = (0, 0, \dots, 1)$$

and hence with the determinant $d(A) = 1$. These n basis points belong to S , but not to $z^{-1}S$; therefore

$$m_k(S, A) = 1 \quad (k = 1, 2, \dots, n),$$

which shows that (I) holds with equality.

Proof of Eq. (6). We apply results from Mahler [3]. In the notation of this paper, S is a *convex body of volume* $V = 1$.

Let A be an arbitrary lattice of basis $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$, where

$$\mathbf{x}^{(k)} = (x_{k1}, \dots, x_{kn}) \quad (k = 1, 2, \dots, n)$$

and determinant

$$d(\mathcal{A}) = |D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})|. \quad (7)$$

The general point \mathbf{x} of \mathcal{A} has the form

$$\mathbf{x} = u_1 \mathbf{x}^{(1)} + \dots + u_n \mathbf{x}^{(n)}, \quad \text{where } u_1, \dots, u_n \in f[z].$$

We put

$$\mathbf{u} = (u_1, \dots, u_n)$$

and consider the linear mapping in F^n from \mathbf{u} to \mathbf{x} . In the coordinates of \mathbf{u} the lattice \mathcal{A} is transformed into the lattice \mathcal{A}^* of basis $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n)}$. The mapping from \mathcal{A} to \mathcal{A}^* has the determinant $D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})$ which occurs in (7). The function $|\mathbf{x}|$ takes in the new variables u_1, \dots, u_n the form

$$|\mathbf{x}| = \max_{k=1,2,\dots,n} |u_1 x_{1k} + \dots + u_n x_{nk}| = \Psi(\mathbf{u}) \quad \text{say.}$$

Here $\Psi(\mathbf{u})$ is a *convex distance function* in the terminology of Mahler [3], and the corresponding *convex body* K of all points \mathbf{u} satisfying

$$\Psi(\mathbf{u}) \leq 1$$

has by my paper the *volume*

$$V = |D(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})|^{-1} = d(\mathcal{A})^{-1}.$$

Now, by Section 9 of [3], there exist n points $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(n)}$ of determinant 1 with coordinates in $f[z]$ (i.e., these points form a basis of \mathcal{A}^*) such that

$$\prod_{k=1}^n \Psi(\mathbf{u}^{(k)}) = V^{-1} = d(\mathcal{A}). \quad (8)$$

The coordinates of these points are, say,

$$\mathbf{u}^{(k)} = (u_{k1}, \dots, u_{kn}) \quad (k = 1, 2, \dots, n).$$

We derive from them the points

$$\mathbf{X}^{(k)} = u_{k1} \mathbf{x}^{(1)} + \dots + u_{kn} \mathbf{x}^{(n)} \quad (k = 1, 2, \dots, n)$$

which form a basis of the original lattice \mathcal{A} .

By (8), this basis now satisfies the equation

$$|\mathbf{X}^{(1)}| |\mathbf{X}^{(2)}| \dots |\mathbf{X}^{(n)}| = d(\mathcal{A}).$$

Here each of the factors $|\mathbf{X}^{(1)}|, |\mathbf{X}^{(2)}|, \dots, |\mathbf{X}^{(n)}|$ is an integral power of 2. If

now $d(A) \leq 2^{n-1}$, then at least one factor $|X^{(k)}|$ does not exceed 1 and hence $X^{(k)}$ lies in S . But $X^{(k)}$ is certainly distinct from 0 ; therefore A is not S -admissible.

It follows that A only then can be S -admissible if $d(A) \geq 2^n$, giving the assertion because the lattice of basis $ze^{(1)}, \dots, ze^{(n)}$ and determinant 2^n is S -admissible. This proves the assertion.

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